THEORETICAL INVESTIGATION OF SEMI-INFINITE ICE FLOES IN WATER OF INFINITE DEPTH

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by

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ABSTRACT

The response of semi-infinite ice floes to water waves is analyzed for relatively deep water. If the floe submergence is neglected it is found that a progressive wave is transmitted. The stress produced by this transmitted wave is determined for various floe thicknesses and incident wave lengths. When the submergence is not neglected it is necessary to use a finite difference approach to the solution. Such a solution is attempted and the results and accompanying numerical problems are considered in detail.

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INTRODUCTION

This study continues the effort to analyze the effect of water waves on ice floes. The previous study (Ref. 1) dealt with finite ice floes with negligible depth of submergence. In this work the effect of water waves on semi-infinite ice floes with zero and finite submergence depths is studied.

The first portion of the report is an analysis of the zero submergence problem. The governing equations are established and a closed-form expression for the stress produced by the transmitted wave is obtained. This solution depends upon the roots of a quintic equation, which involve the parameters of the incident wave as well as ice properties and floe thickness. This stress is calculated for various floe thicknesses and incident wave lengths assuming constant ice floe properties. A sample calculation is included. Reflection and transmission coefficients are also evaluated and discussed.

The finite submergence case is then analyzed. Attempts at an analytical solution are discussed and due to the complexity of the equations these are discarded.

Finally, a finite difference approach to the solution is used to determine the stress in the floe near the leading edge. The limitations of computer storage severely restrict the size of the flow field under consideration but a definite indication to the stress pattern is established.

SEMI-INFINITE PLATE NO SUBMERGENCE

Consider a semi-infinite elastic plate of thickness h floating upon water of relatively infinite depth, as shown in Fig. 1.

If linearized theory is used then the free surface portion of the flow field, Region 1, is given by the potential function

$$\text{(1)} \qquad \phi_1 = (B\, e^{-ikx} + R\, e^{ikx})\, e^{-ky}\, e^{i\sigma t}$$

with

$$\text{(2)} \qquad \frac{\partial \eta_1}{\partial t} = -\left.\frac{\partial \phi_1}{\partial y}\right)_{y=0}, \quad \eta_1 = -\frac{1}{g}\left.\frac{\partial \phi_1}{\partial t}\right)_{y=0}$$

and therefore

$$\text{(3)} \qquad \sigma^2 = g\, k$$

where

η_1 = local elevation of profile (ft.)

$|\eta_1|$ = incident wave amplitude (ft.)

L_1 = incident wave length (ft.)

T_1 = incident wave period (sec.)

σ = $2\pi/T_1$

k = $2\pi/L_1$

U = $|\eta_1| g/\sigma$

R = reflection coefficient

B = $i\, U$

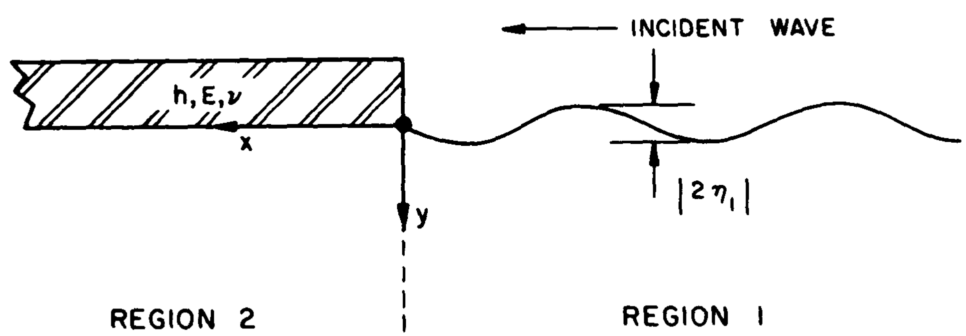


FIGURE 1
SEMI-INFINITE PLATE

The equation for the oscillation of a plate is

$$(4) \qquad D \frac{\partial^4 \eta_2}{\partial x^4} + \rho_s h \frac{\partial^2 \eta_2}{\partial t^2} = p$$

where

η_2 = displacement of the plate

$D = \dfrac{h^3 E_s}{12(1 - \nu^2)}$ flexural rigidity of the plate

h = plate thickness

ρ_s = mass density of plate

p = pressure exerted on the plate by the fluid

ν = Poisson's ratio of the plate $= \frac{1}{3}$

E_s = modulus of elasticity of the plate

Now if Φ_2 is the potential function for Region 2 and if Φ_2 is of the form

$$(5) \qquad \Phi_2 = A_n e^{i\lambda_n x} e^{-\lambda_n y} e^{i\sigma t}$$

then the pressure p as shown in Ref. 1 is given by

$$(6) \qquad p = -\rho_f \left[-\left(\frac{\partial \Phi_2}{\partial t} \right)_{y=0} + g\eta_2 \right]$$

where ρ_f is the fluid density.

If the plate displacement is equal to the water displacement and the velocities correspond, then

$$\frac{\partial \eta_2}{\partial t} = -\left.\frac{\partial \Phi_2}{\partial y}\right)_{y=0} \qquad \text{or} \qquad \eta_2 = -\frac{i}{\sigma}\left.\frac{\partial \Phi_2}{\partial y}\right)_{y=0}$$

Thus, the plate displacement is

$$(7) \qquad \eta_2 = \frac{i}{\sigma}\,\lambda_n A_n\, e^{i\lambda_n x}\, e^{-\lambda_n y}\, e^{i\sigma t}$$

where the λ_n are five roots of the characteristic equation (9), which is obtained by substituting (7) into (4)

$$(8) \qquad -\frac{i}{\sigma}\left\{D\,\frac{\partial^4}{\partial x^4} - \rho_s h \sigma^2 + \rho_f g\right\}\left.\frac{\partial \Phi_2}{\partial y}\right)_{y=0} - i\sigma\rho_f \Phi_2 = 0$$

But

$$\left.\frac{\partial \Phi_2}{\partial y}\right)_{y=0} = -\lambda_n A_n\, e^{i\lambda_n x}\, e^{i\sigma t}$$

therefore the characteristic equation is

$$(9) \qquad \lambda_n^5 + H(1-ak)\lambda_n - Hk = 0$$

$$\text{where} \qquad H = \frac{\rho_f g}{D} \qquad\qquad a = \frac{\rho_s h}{\rho_f}$$

and $k = \sigma^2/g$ for the incident wave.

The roots of the characteristic equation are λ_n . With a, b, c, d all greater than zero the λ_n may be written in the form

λ_o a positive real number

$\lambda_1 = a + bi \qquad \lambda_2 = a - bi$

$\lambda_3 = -c + di \qquad \lambda_4 = -c - di$

but since the potential function Φ_2 must decay with increasing y, λ_3 and λ_4 are not admissible. The potential Φ_2 can be written as

$$(10) \quad \Phi_2 = \left(T e^{-i\lambda_o x - \lambda_o y} + A_1 e^{i\lambda_1 x - \lambda_1 y} + A_2 e^{-i\lambda_2 x - \lambda_2 y}\right) e^{i\sigma t}$$

Now $e^{(-i\lambda_o x - \lambda_o y)}$ represents a progressive wave as x approaches infinity and thus, the term T is related to the transmission coefficient. The terms $A_1 e^{(i\lambda_1 x - \lambda_1 y)}$ and $A_2 e^{(-i\lambda_2 x - \lambda_2 y)}$ are waves which decay both with increasing x and increasing y.

Now for continuity along the line $x = 0$ the potential functions and particle velocities must be identical so that

$$(11) \quad \begin{aligned} \Phi_1 &= \Phi_2 \quad \text{at} \quad x = 0 \\ \frac{\partial \Phi_1}{\partial x} &= \frac{\partial \Phi_2}{\partial x} \quad \text{at} \quad x = 0 \end{aligned}$$

The plate has a free end at $x = 0$ so that the moment and shear must be zero at that point, giving

$$(12) \quad \frac{\partial^2 \eta_2}{\partial x^2} = 0 \text{ at } x = 0 , \qquad \frac{\partial^3 \eta_2}{\partial x^3} = 0 \text{ at } x = 0$$

Now (11) and (12) give four boundary conditions which are sufficient to determine the four unknown coefficients R, T, A_1 and A_2 .

If the subscript $(\ )_1$ refers to the real part of a coefficient and $(\ )_2$ refers to the imaginary part of a coefficient (T_1 is not to be confused with the period of the incident wave, T_1 which is not used explicitly in the analysis), then after considerable algebraic manipulation it is found that

$$
(13)\quad
\begin{aligned}
T_1 &= -2\gamma\,\Delta\, GU/J \\
T_2 &= 2\gamma\,\Delta\, FU/J \\
A_{11} &= \frac{1}{2\Delta}\left[(\gamma_3-\gamma_4)\,T_1+(\delta_3-\delta_4)\,T_2\right] \\
A_{12} &= -\frac{1}{2\Delta}\left[(\delta_3-\delta_4)\,T_1-(\gamma_3-\gamma_4)\,T_2\right] \\
A_{21} &= -\frac{1}{2\Delta}\left[(\gamma_3+\gamma_4)\,T_1-(\delta_3+\delta_4)\,T_2\right] \\
A_{22} &= -\frac{1}{2\Delta}\left[(\delta_3+\delta_4)\,T_1+(\gamma_3+\gamma_4)\,T_2\right] \\
R_1 &= T_1+A_{11}+A_{21} \\
R_2 &= T_2+A_{12}+A_{22}-U
\end{aligned}
$$

where $A_1 = A_{11} + i\,A_{12}$, $A_2 = A_{21} + i\,A_{22}$ etc. and

$$\gamma = \frac{k}{\lambda_o}, \qquad \gamma_n = \mathrm{Re}\left(\frac{\lambda_1}{\lambda_o}\right)^n, \qquad \delta_n = \mathrm{Im}\left(\frac{\lambda_1}{\lambda_o}\right)^n$$

$$\Delta = \gamma_3\gamma_4 + \delta_3\delta_4$$

$$F = \Delta\,(1+\gamma) - \gamma\gamma_4 - \gamma_1\gamma_3 - \delta_1\delta_3$$

$$G = \gamma\delta_3 + \gamma_1\delta_4 - \delta_1\gamma_4$$

$$J = F^2 + G^2$$

In order to find the relation between the transmitted, reflected and input waves define $E\bar{E} = R\bar{R} + T\bar{T}$ where the bar indicates complex conjugate, then

$$E\bar{E} = R_1^{\ 2} + R_2^{\ 2} + T_1^{\ 2} + T_2^{\ 2} \tag{14}$$

Upon substitution of the quantities T_i and R_i from (13) it is found that

$$\frac{E\bar{E}}{U^2} = 1 + \frac{4\gamma}{J}\Big[2\gamma\Delta^2 - \Delta(\gamma\gamma_4 - 1 - \gamma - \gamma_1 + \gamma_4) + \gamma\gamma_4 + \gamma_1\gamma_3 + \delta_1\delta_3\Big] \tag{15}$$

Now $E\bar{E}$ is a positive quantity, being a sum of squares of real numbers, therefore $\frac{E\bar{E}}{U^2}$ exceeds unity. This is an apparent violation of the equation of conservation of energy which apparently would require $E\bar{E} = U^2 = R\bar{R} + T\bar{T}$. But work has also been done in permanently changing the wave length from $L_1 = 2\pi/k$ for the incident waves to $L_o = 2\pi/\lambda_o$ for the transmitted waves and (15) expresses this.

Fig. 2 shows the variation of

$$\epsilon = \frac{(E\bar{E})^{1/2}}{U}, \quad \rho = \frac{(R\bar{R})^{1/2}}{U}, \quad \tau = \frac{(T\bar{T})^{1/2}}{U}$$

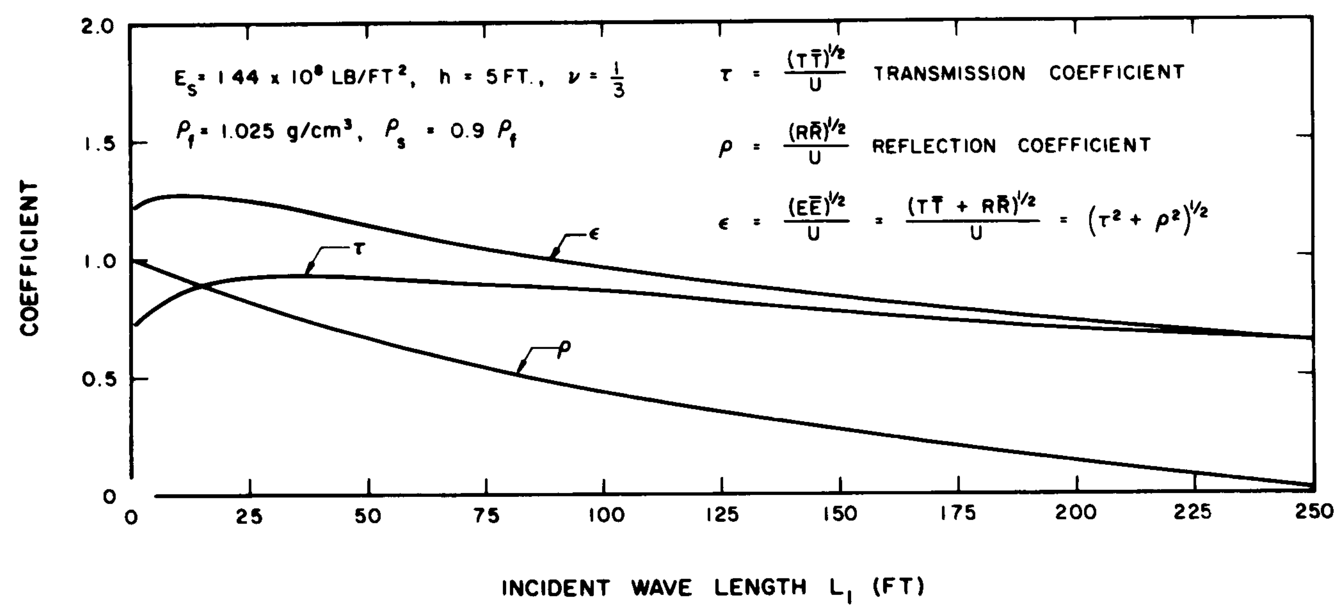


FIGURE 2
REFLECTION AND TRANSMISSION COEFFICIENTS VS. INCIDENT WAVE LENGTH

with incident wave length and plate thickness.

The stress in the plate reaches a maximum value near the leading edge and then decays to a slightly lower value given by the amplitude of the transmitted wave. The peak stress is analogous to a transient and the stress due to the transmitted wave is analogous to a steady state value in the sense that it is independent of x .

The transient stress S_x is found from the expression

$$S_x = \left(C_{1x}^{\ 2} + C_{2x}^{\ 2}\right)^{1/2}$$

where

$$C_{1x} = \frac{E_s h \lambda_o^{\ 3}}{2\gamma_1 \sigma (1-\nu^2)} \left(T_1 P_1 + T_2 P_2\right)$$

$$C_{2x} = \frac{E_s h \lambda_o^{\ 3}}{2\gamma_1 \sigma (1-\nu^2)} \left(-T_1 P_2 + T_2 P_1\right)$$

where

$$P_1 = -\gamma_1 \cos \lambda_o x + e^{-bx}(\gamma_1 \cos ax + \delta_1 \sin ax)$$

$$P_2 = -\gamma_1 \sin \lambda_o x + e^{-bx} \sin ax$$

and the units of S_x will be lb/ft^2 per foot of incident wave amplitude.

The stress due to the transmitted wave S^* is found by allowing x to approach infinity in the preceding expressions

$$S^* = \left(C_1^{\ 2} + C_2^{\ 2}\right)^{1/2}$$

where

$$C_1 = \frac{E_s h \lambda_o^{\ 3}}{2\sigma (1-\nu^2)} \left(-T_1 \cos \lambda_o x - T_2 \sin \lambda_o x\right)$$

$$C_2 = \frac{E_s h \lambda_o^{\,3}}{2\sigma(1-\nu^2)} \left(T_1 \sin \lambda_o x - T_2 \cos \lambda_o x\right)$$

giving finally

$$S^* = \frac{9 E h \lambda_o^{\,3} \gamma \Delta U}{8\sigma} J^{-1/2}$$

for $\nu = 1/3$.

The quantity S^* for various incident wave lengths and plate thicknesses is shown in Fig. 3 .

For comparison with the finite submergence case, the stress amplitude S_x for an incident wave length of 10 feet and plate thickness of 5 feet has been worked out in detail as shown in Fig. 4 .

As an illustrative example, suppose it is desired to find the stress amplitude for a 15 foot thick floe at an incident wave length of 100 feet with an incident wave amplitude of 6 inches. From Fig. 3, S^* for $h = 15$, $L_1 = 100$ is 112 lb/in^2 therefore the desired value of S^* will be

$$S^* = \frac{6}{12} \, 112 = 56 \ \text{lb/in}^2$$

since the stress amplitude graph is normalized to a 1 foot amplitude.

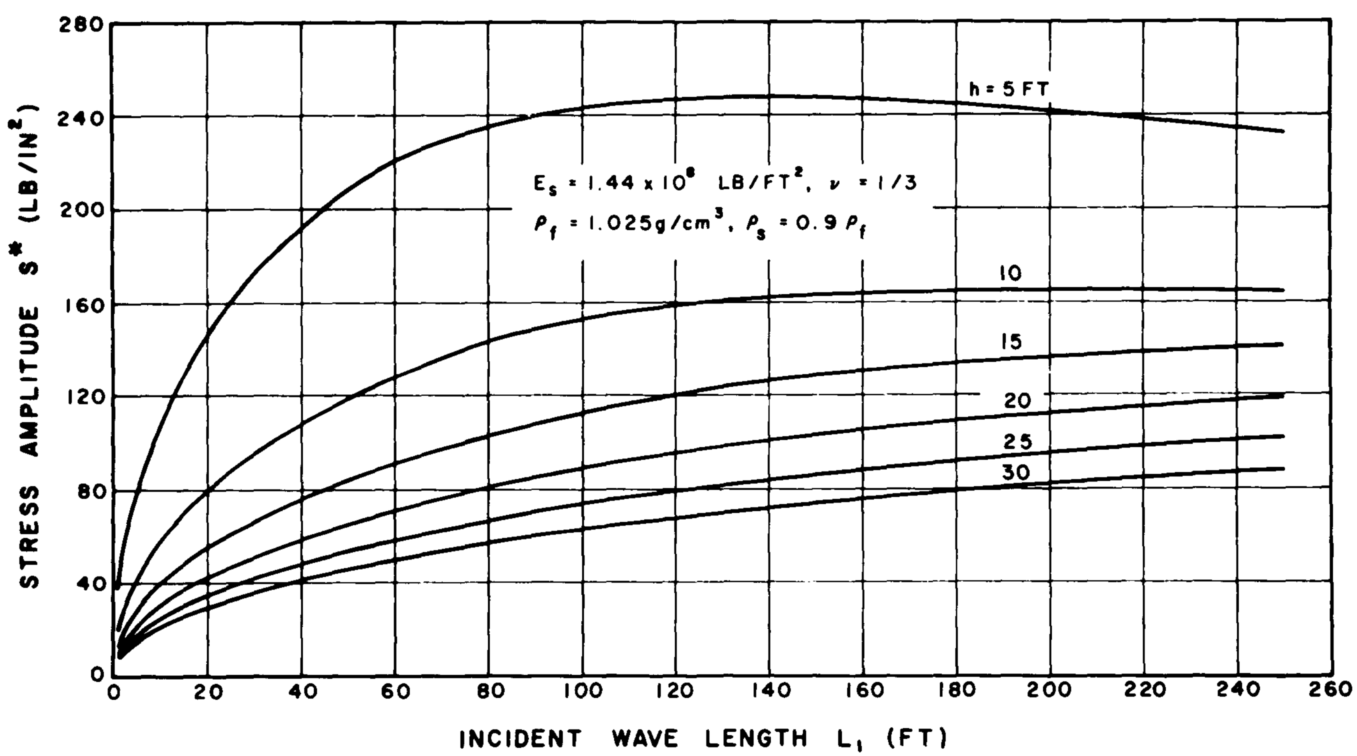


FIGURE 3
STRESS AMPLITUDE VS. INCIDENT WAVE LENGTH

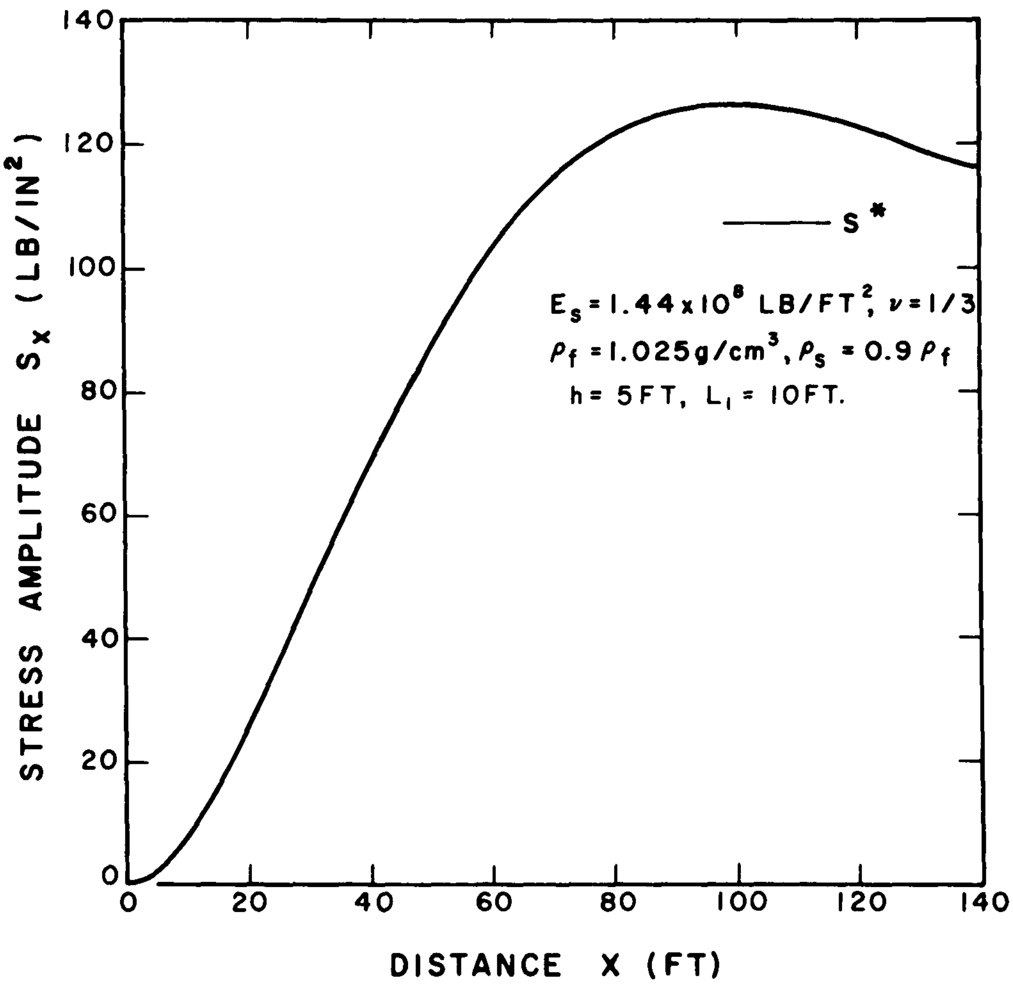


FIGURE 4

STRESS AMPLITUDE IN PLATE (NO SUBMERGENCE)

SEMI-INFINITE PLATE WITH FINITE SUBMERGENCE

We consider a semi-infinite elastic plate submerged in water to a depth "a" below the mean water line of a train of incident harmonic waves. The coordinate y is measured positive downward from the mean water line and the coordinate x is measured parallel to the mean water line from the leading edge of the plate positive in the direction of incoming wave propagation (i.e. positive into the plate).

The free surface portion of the flow field must obey equation (2) of the previous section of this report. It is assumed that this submerged edge of the plate perturbs a relatively small portion of the flow field and therefore that as the distance from this area increases, the general simple harmonic character of the incident, reflected and transmitted waves is maintained.

Ursell (Ref. 2) has shown that if a normal velocity is prescribed on part of a vertical plane (i.e. thin boundary) extending from the surface then the two dimensional problem of determining the motion on either side of the plane as well as beneath it can be solved for the deep water case. The motion is determined by a comparatively simple integral equation. At the lower edge of the plane the velocity becomes infinite due to a mathematical singularity of logarithmic type. Ursell obtains his solution by assuming that the transmitted waves have the same wave length as the incident waves.

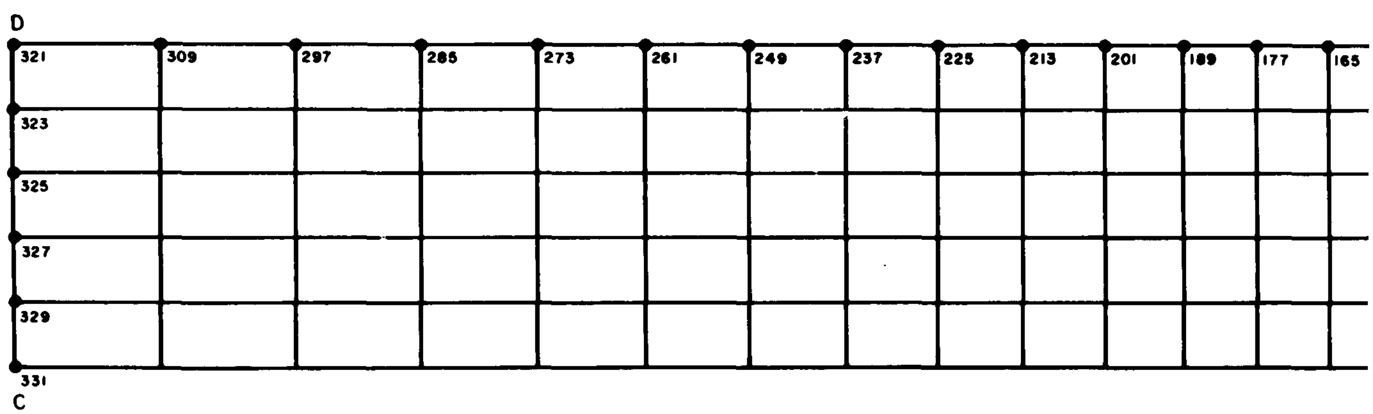
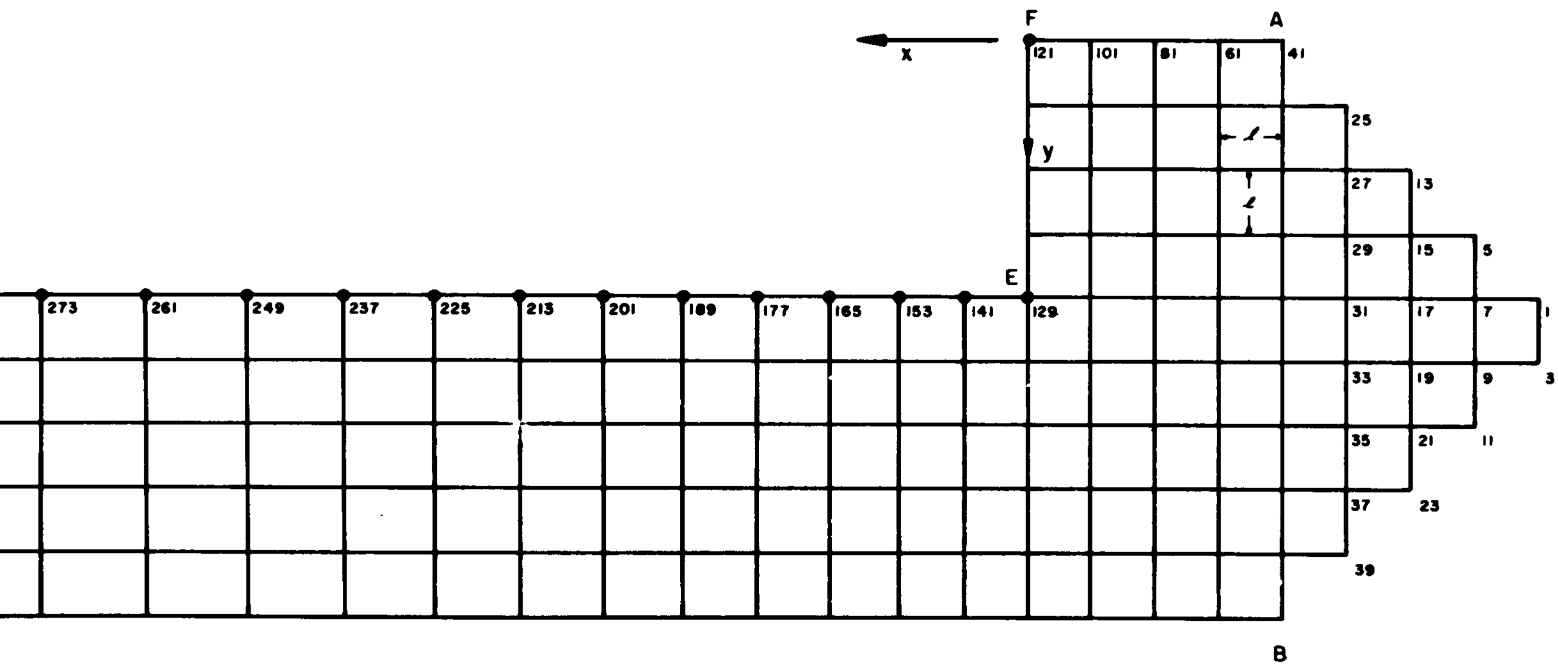


FIGURE 5

FINITE DIFFERENCE APPROXIMATION TO THE FLOW FIELD



ON TO THE FLOW FIELD

An attempt was made to modify Ursell's approach to include the effect of the plate as well as the change in wave length of transmitted waves. In so doing a very complicated set of simultaneous integral equations is obtained. The kernels of these equations involve the fifth order equation for the plate oscillation. The integrals were found to be intractable and hence the analytical approach was discarded in favor of a numerical means of solution.

Since no analytical solution was found for the plate with finite submergence an approximate solution was attempted. The coordinate system is shown in Fig. 5.

The incident waves are given by the potential function

$$(16) \qquad \Phi_1 = \left(iUe^{-ikx} + Re^{ikx} \right) e^{-ky} e^{i\sigma t}$$

The submerged edge of the plate is assumed to perturb a relatively small portion of the flow field, therefore, at some station AB where $x = x_1$, the potential is

$$(17) \qquad \Phi_{AB} = \left(iUe^{-ikx_1} + Re^{ikx_1} \right) e^{-ky} e^{i\sigma t}$$

Now as y increases $\frac{\partial \Phi}{\partial y}$ vanishes, thus, for sufficiently large y

$$(18) \qquad \left. \frac{\partial \phi_{BC}}{\partial y} \right)_{y = y_1} = 0$$

Again for sufficiently large $x = x_2$ the potential ϕ_{CD} which is the asymptotic form of Φ_2 given in equation (10) is assumed to exist in the form

$$(19) \qquad \Phi_{CD} = T e^{-i\lambda_o x} e^{-\lambda_o y} e^{i\sigma t}$$

Along the boundary DE the equation of motion of the plate must hold and this is given by

$$(20) \qquad \left[\frac{\partial^2}{\partial x^4} + H(1-ak) \right] \left(\frac{\partial \Phi_{DE}}{\partial y} \right)_{y=a} + Hk \left(\Phi_{DE} \right)_{y=a} = 0$$

Along the submerged edge of the plate EF

$$(21) \qquad \frac{\partial \Phi_{EF}}{\partial x} = 0$$

since this edge does not move in the x direction. The free surface condition along FA is analagous to equation (2)

$$(22) \qquad \Phi = - \frac{1}{k} \left. \frac{\partial \Phi}{\partial y} \right)_{y=0}$$

Since the plate has a free end at the point x = 0, y = a,

$$(23) \qquad \begin{aligned} \frac{\partial^2}{\partial x^2} \left(\frac{\partial \Phi}{\partial y} \right)_{\substack{x=0 \\ y=a}} &= 0 \\ \frac{\partial^3}{\partial x^3} \left(\frac{\partial \Phi}{\partial y} \right)_{\substack{x=0 \\ y=a}} &= 0 \end{aligned}$$

The field equation is Laplace's Equation

$$\nabla^2 \Phi = 0 \tag{24}$$

which holds everywhere within A B C D E F.

There are now sufficient equations to determine Φ , R and T.

The size of the grid ℓ for a finite difference solution of the problem is unfortunately not arbitrary. The core storage capacity of the presently available digital computer equipment (IBM 7090) limits the size of the field considerably.

The following parameters are used in the computation

$$\begin{aligned}
h &= 5 \text{ ft.} \\
E_s &= 1.44 \times 10^8 \text{ lb/ft}^2 \\
\nu &= 1/3 \\
\rho_f &= 1.025 \text{ gm/cm}^3 \\
\rho_s &= 0.9 \ \rho_f \\
a &= 4.5 \text{ feet} \\
\ell &= \frac{a}{4} = 1.125 \text{ ft.}
\end{aligned}$$

Equal divisions of length ℓ define the nodes in the free-surface portion of the field. There are 10 nodes in the y-direction and at most 8 in the x-direction. The field under the plate is a semi-logarithmic distribution of nodes where the logarithmic scale is in the direction of increasing x and the y divisions are ℓ apart as in the free-surface region. There are 16 nodes in the x-direction and 6 in the y-direction.

The equation for the x-divisions is

$$x = \frac{1.125}{\log 27 - \log 26} \quad \log 27 - \log (27-m) \tag{25}$$

where $0 \leq m \leq 16$ in unit steps.

In order to conserve computer storage and yet include as much of the perturbed flow field as possible, the free-surface portion of the field was stepped as shown in Fig. 5 .

Assuming a second order function for ϕ then the field equation can be written as

$$\begin{aligned} \nabla^2 \Phi = \frac{1}{\alpha(\alpha+\gamma)} \Phi_1 + \frac{1}{2\ell^2} \Phi_2 + \frac{1}{\gamma(\alpha+\gamma)} \Phi_3 + \frac{1}{2\ell^2} \Phi_4 \\ - \left(\frac{1}{\alpha\gamma} + \frac{1}{\ell^2}\right) \Phi_o = 0 \end{aligned} \tag{26}$$

where α and γ and Φ_N are defined in Fig. 6 .

Along the free surface FA the slope condition (22) permits elimination of the ficticious point "1" giving, since $\alpha = \gamma = \ell$,

$$\nabla^2 \Phi_{FA} = \frac{1}{2\ell^2} \left[\Phi_1 + 2\Phi_4 + \Phi_3 - (4-2k\ell)\Phi_o \right] = 0$$

where the Φ's have the same relations to Φ_o as shown in Fig. 6 .

Along the lower boundaries AB, BC, CD Eqn. (24) the field equation becomes respectively,

$$\frac{1}{2\ell^2} \left(\Phi_2 + \Phi_3 + \Phi_4 - 4\Phi_o\right) = \frac{1}{2\ell^2} \quad \Phi_1 \tag{28}$$

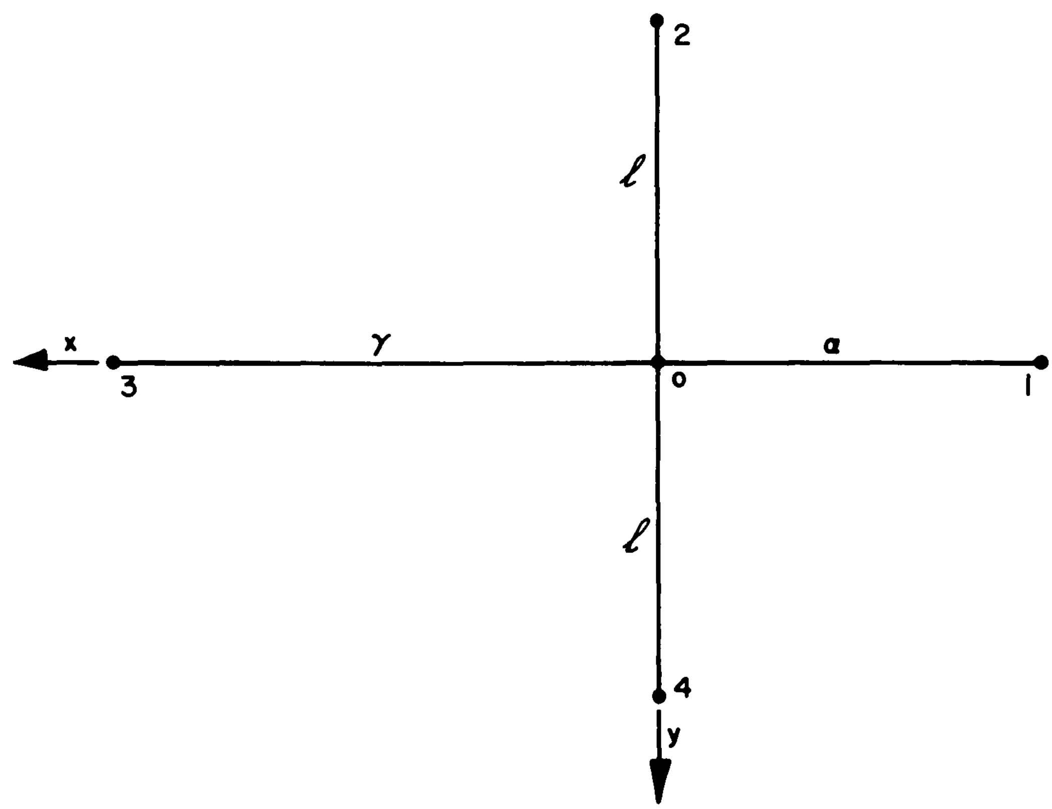


FIGURE 6
FINITE DIFFERENCE SCHEME FOR THE FIELD EQUATION

where Φ_1 is given by (17)

$$(29)\quad \nabla^2\Phi_{BC} = \frac{1}{\alpha(\alpha+\gamma)}\Phi_1 + \frac{1}{\ell^2}\Phi_2 + \frac{1}{\gamma(\alpha+\gamma)}\Phi_3 - \left(\frac{1}{\alpha\gamma} + \frac{1}{\ell^2}\right)\Phi_o = 0$$

$$(30)\quad \nabla^2\Phi_{CD} = \frac{2}{\alpha(\alpha+\gamma)}\Phi_1 + \frac{1}{2\ell^2}\Phi_2 + \Phi_4 - \left(\frac{1}{\alpha\gamma} + \frac{1}{\ell^2}\right)\Phi_o = 0$$

Along the boundary DE the high order differential equation is expressed in finite difference form using the scheme shown in Fig. 7 .

Defining $Z_N = x_N - x_1$ and $y' = y - 4\ell$ then, if

$$V_N = \left.\frac{\partial\Phi}{\partial y}\right)_{\substack{Z_N \\ y'=0}}$$

and V_N is a fourth order equation in x

$$(31)\qquad V_N = \frac{\Phi_{N+6} - F_N}{2\ell} = \sum_{0}^{4} M_n Z_N{}^n$$

The constants M_n can be evaluated by curve fitting as before. It is found that

$$(32)\qquad M_4 = v_1 V_1 - v_2 V_2 + v_3 V_3 - v_4 V_4 + v_5 V_5$$

where

$$v_1 = \frac{1}{Z_2 Z_3 Z_4 Z_5}$$

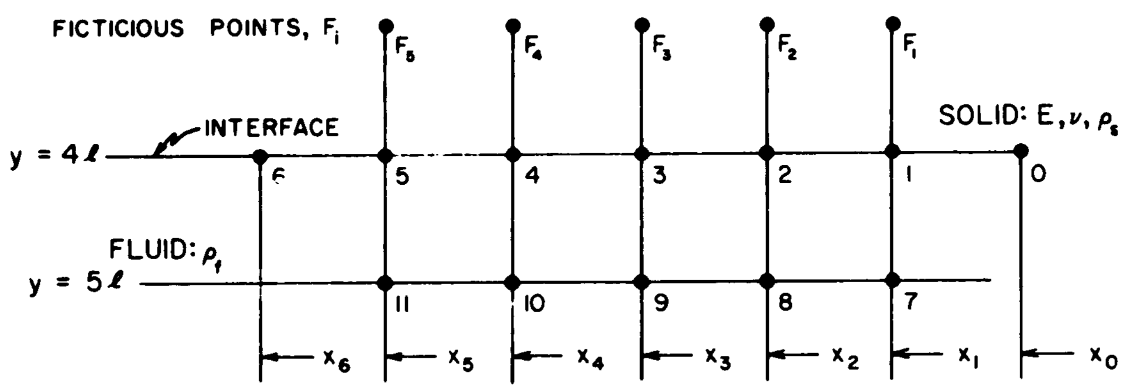


FIGURE 7
DETAIL FOR EQUATION (20) AT POINT "1" (SCHEMATIC)

$$v_2 = \frac{1}{Z_2(Z_5 - Z_2)(Z_4 - Z_2)(Z_3 - Z_2)}$$

$$v_3 = \frac{1}{Z_3(Z_5 - Z_3)(Z_4 - Z_3)(Z_3 - Z_2)}$$

$$v_4 = \frac{1}{Z_4(Z_5 - Z_4)(Z_4 - Z_3)(Z_4 - Z_2)}$$

$$v_5 = \frac{1}{Z_5(Z_5 - Z_4)(Z_5 - Z_3)(Z_5 - Z_2)}$$

Now using the field equation $\nabla^2 \Phi = 0$ the ficticious points F_N in (31) can be eliminated. For example, at point "N"

$$\nabla^2 \Phi_N = \frac{\Phi_{N-1}}{(X_N - X_{N-1})(X_{N+1} - X_{N-1})} + \frac{1}{2\ell^2} F_N$$

$$- \left[\frac{1}{\ell^2} + \frac{1}{(X_N - X_{N-1})(X_{N+1} - X_N)} \right] \Phi_N$$

$$+ \frac{1}{2\ell^2} \Phi_{N+6} + \frac{\Phi_{N+1}}{(X_{N+1} - X_N)(X_{N+1} - X_{N-1})} = 0$$

so that

$$V_N = \frac{1}{2\ell} \left\{ \frac{\Phi_{N-1}}{(X_N - X_{N-1})(X_{N+1} - X_{N-1})} \right.$$

(33) $$\left[\frac{1}{\ell^2} + \frac{1}{(X_{N-1} - X_N)(X_{N+1} - X_N)} \right] \Phi_N$$

$$\left. + \frac{1}{\ell^2} + \Phi_{N+6} + \frac{\Phi_{N+1}}{(X_{N+1} - X_N)(X_{N+1} - X_{N-1})} \right\}$$

Finally, equation (20) becomes

$$(34) \qquad 24\, M_4 + H(1-ak)\, V_N + Hk\, \Phi_N = 0$$

Similar equations express the conditions at the free end (23). Along the boundary EF using (21)

$$(35) \qquad \frac{1}{2\ell^2}\left(2\Phi_1 + \Phi_2 + \Phi_4 - 4\Phi_o\right) = 0$$

The system illustrated in Fig. 5 consists of 332 unknown values of Φ where $\phi_{2n-1} = \mathrm{Re}\,(\Phi)$ and $\phi_{2n} = \mathrm{Im}\,(\Phi)$, plus four additional unknowns: The transmission coefficient $T = T_1 + i\,T_2$ and reflection coefficient $R = R_1 + i\,R_2$. There are 336 equations in the system; 332 statements at node points and four additional equations for the free-end conditions at $x = 0$.

The value of the stress amplitude S_x is found from the equation

$$(36) \qquad S_x = s\,(S_1^{\,2} + S_2^{\,2})^{1/2}$$

where

$$S_1 = \mathrm{Re}\left\{ \frac{\partial^2 \eta_2}{\partial x^2}\, e^{i\sigma t} \right\}$$

$$S_2 = \mathrm{Im}\left\{ \frac{\partial^2 \eta_2}{\partial x^2}\, e^{i\sigma t} \right\}$$

and $s = \dfrac{E_s\, h}{2(1-\nu^2)}$

Values of the stress S_x are shown in Fig. 8, where the corresponding values determined for the same plate without submergence are also shown.

In general the solution of the finite difference formulation of the problem of non-zero submergence involves the following steps:

1. Establish the finite difference grid and ide...ify the nodes.
   a. Choose ℓ .
   b. Choose the semi-logarithmic law analagous to equation (25) for the variation of x beneath the plate.
2. Set-up the linear equations at each node. There will be two equations at each node (except at the plate corner E). One equation will be for the Real part of Φ and the other for the imaginary part.
   a. Use Eqn. (26) at each interior node.
   b. Use Eqn. (27) along the boundary FA.
   c. Use Eqn. (28) along the boundary AB.
   d. Use Eqn. (29) along the boundary BC.
   e. Use Eqn. (30) along the boundary CD.
   f. Use Eqn. (34) along the boundary DE.
   g. Use Eqn. (35) along the boundary EF.
   h. At point E state equations (23) using M_2 and M_3 from Eqn. (31).

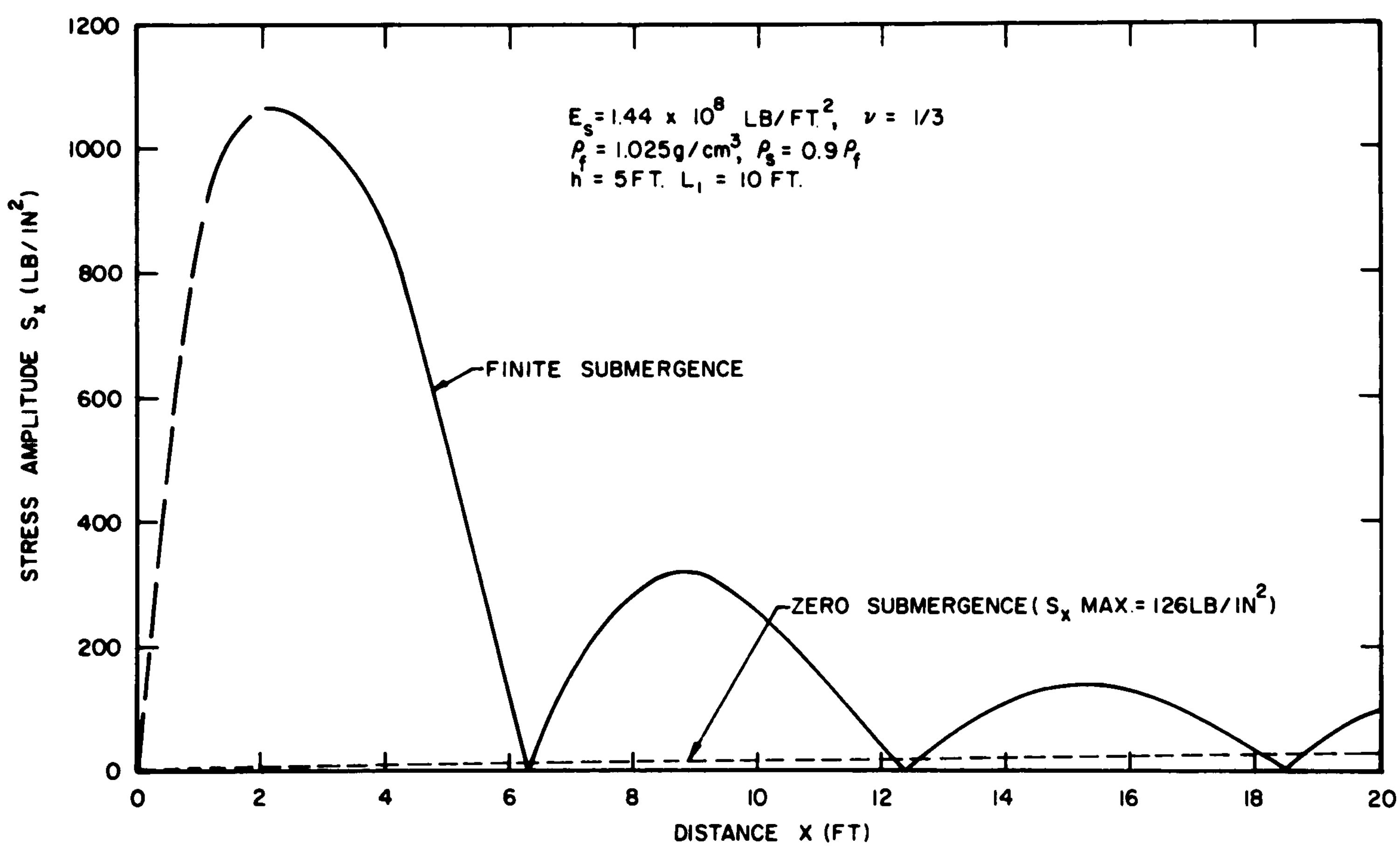


FIGURE 8
STRESS AMPLITUDE IN PLATE

3. Band the matrix and solve. The computer routine used for this report was "Linear Equation Solver of Banded Matrices", SHARE Distribution Number 990 CORR/1490 F4RWLE4F from the SHARE library.

4. Using the solution for $\mathrm{Re}\,\Phi$ and $\mathrm{Im}\,\Phi$ determine $\frac{\partial \Phi}{\partial y}$ from Eqn. (33), then finally the stress will be given by Eqn. (36).

ANALYSIS OF RESULTS AND CONCLUSIONS

The interaction of progressive waves with a semi-infinite ice floe has been investigated wherein the ice floe is considered equivalent to an elastic sheet. In this investigation two approaches have been used; neglect of the finite submergence of the floe and consideration thereof. In the case of the former approach, a mathematical method is developed wherein a closed form analytic solution for the ice floe response may be obtained. The results of the mathematical analysis show that a portion of the incident wave energy is reflected while some energy is transmitted beneath the ice in the form of a progressive wave whose frequency is dependent on both the characteristics of the ice and the incident wave length. The partition of wave energy into transmitted and reflected energy is represented by potential function coefficients denoted as transmission and reflection coefficients. The variation of these coefficients as a function of incident wave length for a 5 foot thick floe is shown in Figure 2.

Further, the absolute value of the bending stress in the ice floe due to the transmitted wave is shown in Figure 3 as a function of floe thickness and incident wave length.

A specific example showing the variation of the bending stress with the distance from the free edge of the ice floe is shown in Figure 4. The parameters chosen for the calculation were an ice thickness of 5 feet and an incident wave length of 10 feet (also a unity amplitude for the incident wave height). When observing the stress variations depicted in Figures 3 and 4 certain salient points should be observed. First the stresses shown represent "absolute" values. However, the variation of the ice response is harmonic in time. Hence, at a given location in the ice, it is clear that the stress values shown in these figures alternate between the positive value shown and the negative of this value. Secondly, in referring to Figure 4 which shows the stress variation as a function of position from the free edge of the floe, it is important to realize that the "transient" stress values (i.e. those values close to the free edge) are within about 10 per cent of the "steady state" stress values (at positions far removed from the free edge). Also note that the transient effect exists for distances greater than 10 wave lengths of the incident wave from the free edge. Although only one specific example has been calculated for the transient response, this example is important from a comparison standpoint with the numerical example worked for the same floe but where the finite submergence has been included.

Turning now to the numerical example, some comment on the choice of parameters and the mathematics behind the results are in order.

It is clear that the choice of an input wave length of 10 feet and an ice thickness of 5 feet are physically unrealistic (Ref. 3). The necessity for the choice of parameters is due to the computational procedure that we have chosen and not to a limitation imposed by the mathematical model. In order to include the transient effect of the submerged edge of the floe, the finite difference grid should extend several incident wave lengths in either direction from the edge of the floe. Further, to decay the effect of the transient in the vertical direction, the grid should extend several incident wave lengths vertically downward from the edge of the floe. The decay of the transient in all cases may be shown to be of such a form that the longer the incident wave length, the larger must be the region investigated around the submerged edge of the floe. Conversely, as the incident wave length becomes shorter, this region diminishes in extent. Since the size of this region greatly influences the number of grid nodal points investigated and hence the number of simultaneous field and boundary equations (i.e. the size of the solution matrix), it is clear that the solution matrix is of a minimum size for a short wave length.

For the particular incident wave length chosen for the numerical investigation conducted in this report (L = 10 ft.), we obtained a solution matrix corresponding to a total of 336 simultaneous equations and unknowns which correspond to 166 grid nodal points. The particular type of matrix involved in this problem is a banded matrix with a band width of about 80. Such a matrix can be solved by the IBM 7090 using core or immediate access storage. However, this represents the practical limit of the immediate access storage. Hence, an increase in matrix size or equivalently, an increase in input wave length would exceed this storage which would then necessitate the use of tape units. Although such a computational procedure is feasible, the computer run time could easily increase by an order of magnitude or more from the run time using only immediate access core storage. Unfortunately, we were limited in this investigation to a small computer budget. Hence the problem investigated was necessarily tailored so the resulting matrix solution did not exceed core storage. In the event that the computer run time could economically cease to be a governing criterion then longer incident wave lengths and other ice floe thicknesses could be investigated. We feel that this would be valuable from several standpoints.

First, the influence of the incident wave length and ice floe thickness on the ice floe response could better be understood, both from a fundamental standpoint and in comparison to the neglect of the finite floe submergence. Secondly, and of equal importance mathematically, certain computational assumptions used in the present analysis could be more fully investigated. These computational assumptions are outlined as follows:

1. In transforming the Laplacian field equations to finite difference form, second order central difference formulas were developed. However, along the ice-fluid interface it was necessary to switch to higher order forward differences for the fourth order boundary condition. This technique should be verified by increasing the order of the finite difference formula used in the field equations to the same order as the

boundary condition equations. Such an approach would increase the width of the banding in the solution matrix and hence necessitate the use of tape transfer in performing the computer computation.

2. There was no check run on the validity of the choice of grid size. Hence a computational run should be performed on the basis of a smaller size grid. Again, such a process would require the use of tape transfer.

3. Finally, the effect of the region size investigated and the accuracy with which the computer satisfies the assemblage of simultaneous equations should be ascertained.

The first two points, as well as the effect of region size, can most simply be answered by performing multiple solutions; varying certain parameters with each solution, and comparing results in each case. For instance, in the case of the effect of field size, all other parameters being fixed, the field could be increased by varying amounts and the solution for each case compared. When an optimum field size is reached, the solution at any point within the region investigated should not further vary as the field size is increased.

The accuracy of computer solution can only be checked by substituting the answers back into the original equations and investigating the significance of the error between the computed equation value and the input equation value. Such a procedure involves the analysis of solution stability and may be investigated through the use of known techniques.

However, the question raised as to the accuracy of solution will only effect the quantative results of the present analysis. We are, however, in a position to discuss qualitatively the effect of finite submergence of the floe for the numerical example included in this report. A complete knowledge of this effect must naturally await a detailed analysis of the influence of input wave length and floe thickness and an experimental substantiation of the mathematical model used.

The significant qualitative results of the numerical example worked, especially in comparison to the results obtained from the zero submergence case, are listed below:

1. Comparing the stress amplitude variation in Figures 4 and 8 (zero submergence and finite submergence respectively), it may be observed that the transient stress is about an order of magnitude higher near the leading edge of the floe when finite submergence is included. This result is not surprising when it is considered that the barrier caused by the submerged front of the floe severely alters the local flow pattern, which, in turn, highly perturbs the local velocity characteristics inducing a pressure peak close to the floe's leading edge. However, the barrier also tends to greatly reflect the incident wave energy which leaves less energy to be transmitted in the form of progressive waves beneath the floe. Indeed, a

comparison of Figures 4 and 8 tends to substantiate this in that the "steady state" stress appears to be of a lower magnitude in the case of finite submergence than when the submergence is neglected.

2. The transition to the "steady state" response of the floe appears to be much more rapid in the case of finite submergence. In fact, the transient condition appears to die out within several wave lengths of the incident wave, indicating that wave length conversion and energy reflection is taking place very near the edge of the floe.

The above observations lead one to conclude that breakup of the floe might well be a progressive process; initial edge fractures which create new boundaries which, in turn, lead to new edge fractures, etc. Moreover, and of prime importance, is the indication that the magnitude of the steady state stress is less in the case of finite submergence. This implies the possibility that a conservative estimate of the bending stresses in the floe may be obtained on the basis of the relatively simple zero submergence approximation. The justification of this must naturally await more refined and complete analysis.

In NESCO's opinion, it is important to further pursue the mathematical investigation in the directions stated above. However, it is also very important to obtain direct experimental verification of the mathematical model. Hence, due to the high computational costs attendant to a rigorous mathematical extension, we feel that the next logical step in studying the response of ice floes to wave excitation is to obtain direct experimental evidence via field observations. In particular, we urge that studies be made of ice floe deflections as a function of time and location in the floe. A properly designed study of this sort will yield the period of the waves present in the area, the magnitude of stresses involved, their spatial variation, and the dynamic strength characteristics of the ice. Subsidiary studies should be conducted on the importance of floe thickness variations and the range of variation of incident wave lengths and wave amplitudes. A careful analysis of such studies will then indicate the proper direction to proceed mathematically and hence justify and minimize the expenses involved.

A detailed proposal of such an experimental study is presently being prepared by NESCO. This proposal will be submitted in the near future.

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LIST OF SYMBOLS

a	depth of submergence
g	acceleration of gravity
h	floe thickness
k	circular wave-number
ℓ	grid size in finite difference scheme
n	summation index or exponent
p	pressure exerted by fluid on plate
s	stress factor
t	time
v_N	factors in difference expression
x or x_N	dimension measured parallel to surface (positive into floe)
y	depth dimension (positive downward)
z_N	incremental distance in x-direction
A_n	constants in potential function
B	related to incident wave amplitude
C_n	factors in stress expression
D	modulus of rigidity of plate
E_s	modulus of elasticity of plate
E	related to energy content of system
F, G, J	factors in stress expressions
F_N	ficticious point in finite difference scheme
H	factor in characteristic equation

L_o	transmitted wave length	
L_1	incident wave length	
M_N	factor in characteristic equation	
P_1, P_2	factors in stress expression	
R	reflected energy factor	
S^*	"steady state" stress amplitude	
S_x	stress amplitude at station x	
T	incident energy factor	
T_1	incident wave period	
U	incident wave amplitude factor	
V_N	first derivative in y-direction along interface	
N	number of node in finite difference scheme	
γ	ratio of incident wave-number to progressive wave-number	
γ_n	$\mathrm{Re}\left(\frac{\lambda_1}{\lambda_o}\right)^n$	(see definition of λ_1)
δ_n	$\mathrm{Im}\left(\frac{\lambda_1}{\lambda_o}\right)^n$	(see definition of λ_1)
Δ	$\gamma_3\gamma_4 + \delta_3\delta_4$	
ϵ	energy coefficient	
η_1	incident wave profile	
$\lvert\eta_1\rvert$	incident wave amplitude	
η_2	wave profile along interface	
λ_o	progressive wave-number	
λ_1	complex root of characteristic equation	
ν	Poisson's ratio for plate	

Φ	complex potential function
ϕ_{2N-1}	Re $\left(\Phi_{2N-1}\right)$
ϕ_{2N}	Im $\left(\Phi_{2N-1}\right)$
ρ	reflection coefficient
ρ_s	mass density of plate
ρ_f	mass density of fluid
σ	circular frequency
τ	transmission coefficient
∇^2	Laplacian operator
$\overline{(\ )}$	complex conjugate
$(\ )_1$	real part
$(\ )_2$	imaginary part

U. S. Naval Civil Engineering Laboratory, Port Hueneme, California
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1. Ice
2. Stress
3. Water Waves
4. Partial Differential Equations
5. Finite Differences

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